

HYPER-PARAHHERMITIAN MANIFOLDS WITH TORSION

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ABSTRACT. Necessary and sufficient conditions for the existence of a hyper-parahermitian connection with totally skew-symmetric torsion (HPKT-structure) are presented. It is shown that any HPKT-structure is locally generated by a real (potential) function. An invariant first order differential operator is defined on any almost hyper-paracomplex manifold showing that it is two-step nilpotent exactly when the almost hyper-paracomplex structure is integrable. A local HPKT-potential is expressed in terms of this operator. Examples of (locally) invariant HPKT-structures with closed as well as non-closed torsion 3-form on a class of (locally) homogeneous hyperparacomplex manifolds (some of them compact) are constructed.

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1. INTRODUCTION

We study the geometry of structures on a differentiable manifold related to the algebra of paraquaternions together with a naturally associated metric which is necessarily of neutral signature. This structure leads to the notion of (almost) hyper-paracomplex and hyper-parahermitian manifolds in dimensions divisible by four. These structures are also attractive in theoretical physics since some of them play a role in string theory [27, 18, 6, 19] and integrable systems [9].

Hyper-parahermitian geometry may be interpreted as the indefinite analog of hyper-hermitian geometry, but there are important differences. We provide hyper-parahermitian versions of many local and some global results for hyper-hermitian manifolds, specially we adopt the hypercomplex constructions of [16, 14, 4] (but see also [30, 24, 25, 33]).

We treat integrable almost hyper-parahermitian structures, which admit compatible linear connections with totally-skew symmetric torsion, briefly HPKT-structure. It is known that in dimension 4, the conformal structure of neutral signature determined by a hyper-paracomplex structure is necessarily anti-self-dual [1, 18, 21]. We show that the corresponding conformal hyper-parahermitian structure is an HPKT-structure. In higher dimensions, we find necessary and sufficient conditions for the existence of a HPKT-structure in terms of the exterior derivative of the three Kähler forms. We give a holomorphic characterization and show uniqueness of the HPKT connection.

To illustrate the subtleties of HPKT we use some homogeneous examples and their compact factors found in [20]. In particular, we show the existence of an invariant HPKT-structure with closed torsion 3-form on the simple Lie groups $SU(m, m - 1)$, $m > 1$, associated to the biinvariant Killing-Cartan form of neutral signature on $SU(m, m - 1)$. In contrast, the HPKT-structures for the hyper-paracomplex structures on the simple Lie groups $SL(2m - 1, \mathbb{R})$, $m > 1$ obtained in [20] have no compatible biinvariant metric. They may be associated to (a class of) invariant metrics of neutral signature, which however have non-closed torsion forms.

We show that any HPKT-structure is locally generated by a real (potential) function following the ideas developed in [4]. To this end, using Salamon's idea from the quaternionic case (see [30]), we define an invariant first order differential operator D , the hyper-paracomplex operator, on an almost hyper-paracomplex manifold and we show that it is 2-step nilpotent exactly when the almost hyper-paracomplex structure is integrable. Then we obtain the local existence of HPKT-potential by proving the local D -exactness of certain D -closed 2-forms.

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2. HYPERPARAKÄHLER CONNECTION WITH TORSION

Both quaternions H and paraquaternions \tilde{H} are real Clifford algebras, $H = C(2, 0)$, $\tilde{H} = C(1, 1) \cong C(0, 2)$. In other words, the algebra \tilde{H} of paraquaternions is generated by the unity 1 and the generators J_1, J_2, J_3 satisfying the *paraquaternionic identities*,

$$(2.1) \quad J_1^2 = J_2^2 = -J_3^2 = 1, \quad J_1 J_2 = -J_2 J_1 = J_3.$$

We recall the notion of almost hyper-paracomplex manifold introduced by Libermann [23]. An *almost quaternionic structure of the second kind* on a smooth manifold consists of two almost product structures J_1, J_2 and an almost complex structure J_3 which mutually anti-commute, i.e. these structures satisfy the paraquaternionic identities (2.1). Such a structure is also called *complex product structure* [3, 2].

An *almost hyper-paracomplex structure* on a $4n$ -dimensional manifold M is a triple $\tilde{H} = (J_a), a = 1, 2, 3$, where $J_a, a = 1, 2$ are almost paracomplex structures $J_a : TM \rightarrow TM$, and $J_3 : TM \rightarrow TM$ is an almost complex structure, satisfying the paraquaternionic identities (2.1). We note that on an almost hyper-paracomplex manifold there is actually a 2-sheeted hyperboloid worth of almost complex structures: $S_1^2(-1) = \{c_1 J_1 + c_2 J_2 + c_3 J_3 : c_1^2 + c_2^2 - c_3^2 = -1\}$ and a 1-sheeted hyperboloid worth of almost paracomplex structures: $S_1^2(1) = \{b_1 J_1 + b_2 J_2 + b_3 J_3 : b_1^2 + b_2^2 - b_3^2 = 1\}$.

When each $J_a, a = 1, 2, 3$ is an integrable structure, \tilde{H} is said to be a *hyper-paracomplex structure* on M . Such a structure is also called sometimes *pseudo-hyper-complex* [9]. Any hyper-paracomplex structure admits a unique torsion-free connection ∇^{CP} preserving J_1, J_2, J_3 [3, 2] called *the complex product connection*.

The Nijenhuis tensor N_a of J_a is defined by

$$(2.2) \quad N_a(X, Y) = [J_a X, J_a Y] + J_a^2[X, Y] - J_a[J_a X, Y] - J_a[X, J_a Y].$$

It is well known that the structure J_a is integrable if and only if the corresponding Nijenhuis tensor N_a vanishes, $N_a = 0$.

In fact an almost hyper-paracomplex structure is hyper-paracomplex if and only if any two of the three structures $J_a, a = 1, 2, 3$ are integrable due to the existence of a linear identity between the three Nijenhuis tensors [21, 7]. In this case all almost complex structures of the two-sheeted hyperboloid $S_1^2(-1)$ as well as all paracomplex structures of the one-sheeted hyperboloid $S_1^2(1)$ are integrable.

A *hyperparahermitian metric* is a pseudo Riemannian metric which is compatible with the (almost) hyperparacomplex structure $\tilde{H} = (J_a), a = 1, 2, 3$ in the sense that the metric g is skew-symmetric with respect to each $J_a, a = 1, 2, 3$, i.e.

$$(2.3) \quad g(J_1 \cdot, J_1 \cdot) = g(J_2 \cdot, J_2 \cdot) = -g(J_3 \cdot, J_3 \cdot) = -g(\cdot, \cdot).$$

The metric g is necessarily of neutral signature $(2n, 2n)$. Such a structure is called *(almost) hyper-paraHermitian structure*.

Let F_a be the Kähler form associated with the structure $(g, J_a), a = 1, 2, 3$:

$$F_a = g(\cdot, J_a \cdot).$$

The corresponding Lee form is defined by $\theta_a = -\delta F_a \circ J_a^3$. In particular,

$$\theta_a(X) = \sum_{i=1}^{2n} dF_a(e_i, J_a e_i, J_a^2 X),$$

for an orthonormal J_a -adapted basis $\{e_1, \dots, e_{2n}, J_a e_1, \dots, J_a e_{2n}\}$.

If on a hyper-paraHermitian manifold there exists an admissible basis (\tilde{H}) such that each $J_a, a = 1, 2, 3$ is parallel with respect to the Levi-Civita connection or equivalently the three Kähler forms are cosed, $dF_a = 0$ then the manifold is called *hyper-paraKähler*. Such manifolds are also called *hypersymplectic* [15], *neutral hyper-Kähler* [22, 10]. The equivalent characterization is that the holonomy group of g is contained in $Sp(n, \mathbb{R})$ if $n \geq 2$ [34].

For $n = 1$ an (local) almost hyper-paracomplex structure is the same as oriented neutral conformal structure [9, 12, 34, 7]. The existence of a (local) hyper-paracomplex structure is a strong condition since the integrability of the (local) almost hyper-paracomplex structure implies that the corresponding neutral conformal structure is anti-self-dual [1, 18, 21]. The necessary and sufficient condition for the integrability of an (local) almost hyper-paracomplex structure in dimension four is the coincidence of the three Lee forms, $\theta_1 = \theta_2 = \theta_3$ [21].

We use the following notations: For any r -form ω we define $J_a \omega(X_1, \dots, X_r) := (-1)^r \omega(J_a X_1, \dots, J_a X_r)$, $a = 1, 2, 3$ and the operators $d_\alpha \omega := -J_\alpha d J_\alpha \omega$, $\alpha = 1, 2$, $d_3 \omega := (-1)^r J_3 d J_3 \omega$. In particular $d_a F_a = -dF_a(J_a, J_a, J_a)$, $a = 1, 2, 3$.

We consider the (para) complex operators

$$\begin{aligned} \partial_\alpha &= \frac{1}{2} (d + \epsilon d_\alpha), \quad \bar{\partial}_\alpha = \frac{1}{2} (d - \epsilon d_\alpha), \quad \epsilon^2 = 1, \quad \alpha = 1, 2 \\ \partial_3 &= \frac{1}{2} (d + i d_3), \quad \bar{\partial}_3 = \frac{1}{2} (d - i d_3), \quad i^2 = -1. \end{aligned}$$

In particular, a complex function $f = u + iv$ is holomorphic with respect to the complex structure J_3 iff $\bar{\partial}_3 f = 0$ while a paracomplex function $h = u + \epsilon v$ is paraholomorphic with respect to the paracomplex structure $J_\alpha, \alpha = 1, 2$ iff $\bar{\partial}_\alpha h = 0$.

Definition 2.1. A hyperparahermitian metric g is *hyperparaKähler with torsion* (briefly HPKT) if there exists a linear connection ∇ preserving the hyperparacomplex structure whose torsion tensor T^∇ is totally skew-symmetric i.e.

(2.4)

$$\nabla g = \nabla J_1 = \nabla J_2 = \nabla J_3 = 0, \quad T^\nabla(X, Y, Z) := g(T^\nabla(X, Y), Z) = -T^\nabla(X, Z, Y).$$

If the torsion 3-form T^∇ is closed, $dT^\nabla = 0$, then the HPKT-metric is called *strong HPKT metric*.

A connection satisfying condition (2.4) will be called briefly *HPKT-connection*.

Theorem 2.2. Let (M, g, J_1, J_2, J_3) be a hyperparahermitian manifold. The following conditions are equivalent:

- (1) (M, g, J_1, J_2, J_3) admits a HPKT-connection;
- (2) The following equalities hold

$$(2.5) \quad d_1 F_1 = d_2 F_2 = d_3 F_3.$$

In this case the HPKT connection is uniquely determined by the torsion

$$(2.6) \quad T^\nabla = d_1 F_1 = d_2 F_2 = d_3 F_3.$$

In particular, the three Lee forms coincide, $\theta_1 = \theta_2 = \theta_3 = \text{tr}_g T^\nabla$.

Proof. The required connection is the unique Bismut connection determined by Gauduchon [13] (see also [11]) in the hermitian case and by Ivanov-Zamkovoy [21] in the parahermitian case due to the compatibility condition (2.5). Take the trace in (2.6) to get the last identity. \square

It is known that in dimension four any hyperparahermitian metric is anti-self-dual [1, 18, 21]. The proof of Theorem 6.2 in [21] leads to

Proposition 2.3. *Any hyperparahermitian metric on a hyperparacomplex 4-manifold is HPKT. In particular, the Ricci two-forms of the HPKT-connection all vanish.*

3. HOMOGENEOUS EXAMPLES

A non-trivial class of examples for the differential geometric entities defined in the previous section is provided by certain left-invariant HPKT-structures on (semi) simple Lie groups which were found in [20]. For convenience we reproduce here the explicit description of J_2 and J_3 . We define the (para-) complex structures on Lie algebras and then interpret them as homogeneous almost (para-) complex structures on the corresponding simply connected Lie groups. For brevity we shall sometimes abuse notation and proper definitions, by indicating only the Lie algebras.

3.1. HPKT-structure on $SU(m, m-1)$. The most important example from [20] is the group $SU(m, m-1)$, where the biinvariant Killing form is the neutral HPKT metric.

On the simple Lie algebra $su(m, m-1)$ (of dimension $4m(m-1)$) we define a scalar product

$$(3.7) \quad B(X, Y) \doteq \frac{1}{2} \text{tr}(XY), \quad X, Y \in su(m, m-1)$$

Obviously B is proportional to the Killing form and defines a biinvariant, neutral pseudoriemannian metric on $SU(m, m-1)$. Next we produce a convenient B -orthonormal base of the Lie algebra $su(m, m-1)$. As usual, we denote by $E_j^k \in gl(n)$ the matrix with entry 1 at the intersection of the j -th row and the k -th column and 0 elsewhere. We fix the range of indices

$$(3.8) \quad j = 1, \dots, m-1, \quad j < k < 2m-j.$$

Let \mathfrak{z} be the subspace (abelian subalgebra) of $su(m, m-1)$ generated by the elements

$$i(E_j^j + E_{2m-j}^{2m-j} - 2E_m^m), \quad j = 1, \dots, m-1.$$

Let Z^1, \dots, Z^{m-1} be any orthonormal base of \mathfrak{z} , with respect to the scalar product¹ B . We define

$$(3.9) \quad X^j \doteq i(E_j^j - E_{2m-j}^{2m-j}); \quad Y^j \doteq E_j^{2m-j} + E_{2m-j}^j; \quad W^j \doteq i(E_j^{2m-j} - E_{2m-j}^j).$$

$$(3.10) \quad \begin{aligned} U &= U_j^k \doteq \begin{cases} E_j^k - E_k^j & \text{if } j < k \leq m; \\ E_k^{2m-j} - E_{2m-j}^k & \text{if } m < k < 2m - j. \end{cases} \\ V &= V_j^k \doteq \begin{cases} i(E_j^k + E_k^j) & \text{if } j < k \leq m; \\ i(E_k^{2m-j} + E_{2m-j}^k) & \text{if } m < k < 2m - j. \end{cases} \\ S &= S_j^k \doteq \begin{cases} E_k^{2m-j} + E_{2m-j}^k & \text{if } j < k \leq m; \\ -E_j^k - E_k^j & \text{if } m < k < 2m - j. \end{cases} \\ T &= T_j^k \doteq \begin{cases} i(E_k^{2m-j} - E_{2m-j}^k) & \text{if } j < k \leq m; \\ i(E_j^k - E_k^j) & \text{if } m < k < 2m - j. \end{cases} \end{aligned}$$

The invariant vector fields (generated by) $X^j, Y^j, W^j, Z^j, U_j^k, V_j^k, S_j^k, T_j^k$ give a base of the tangent bundle of $SU(m, m-1)$. We define an almost hyperparacomplex structure by

$$(3.11) \quad \begin{aligned} J_3(Z^j) &\doteq X^j; & J_3(Y^j) &\doteq W^j; & J_2(Z^j) &\doteq W^j, & J_2(X^j) &\doteq Y^j; \\ J_3(U_j^k) &\doteq V_j^k; & J_3(S_j^k) &\doteq T_j^k; & J_2(U_j^k) &\doteq T_j^k; & J_2(V_j^k) &\doteq S_j^k \end{aligned}$$

It is shown in [20] that the structure (3.11) is an integrable hyper-paracomplex structure on $SU(m, m-1)$, $m > 1$ which is compatible with the biinvariant (Killing-Cartan) form of neutral signature B .

Now we observe, that the above construction gives also a strong HPKT-structure on $SU(m, m-1)$. The HPKT-connection is the left-invariant connection ∇ , defined by postulating all left-invariant vector fields to be parallel.

The torsion of the above connection is the Lie bracket and the torsion tensor $T^\nabla(X, Y, Z) = -B([X, Y], Z)$ is a closed 3-form (due to the Jacobi identity). So, we have a strong HPKT-structure on $SU(m, m-1)$ which is flat. The compatible neutral Killing-Cartan metric is Einstein.

Simple Lie groups admit cocompact lattices [8], say Γ . Hence, we obtain a HPKT interpretation of the result proved in [20].

Theorem 3.1. [20] *The compact manifolds $SU(m, m-1)/\Gamma$ admit invariant, flat, strong HPKT-structures. The neutral HPKT-metric is a non-flat Einstein metric induced by the Killing-Cartan form.*

Remark 3.2. The above procedure can be applied to the group $(SL(2m-1, \mathbb{C}))^\mathbb{R}$ (see [20]). Thus we obtain invariant strong and flat HPKT-structures on the compact manifolds $(SL(2m-1, \mathbb{C}))^\mathbb{R}/\Gamma$.

¹ B is obviously negative definite on \mathfrak{z} .

3.1.1. *A non-strong HPKT-structure on $SU(2,1)$.* We equipped the 8-dimensional simple Lie group $SU(2,1)$ with a strong and flat left-invariant HPKT-structure induced by the left-invariant hyper-paracomplex structure (3.11) and the Killing-Cartan form. We show below that a similar² hyper-paracomplex structure supports left-invariant HPKT-structure which is not strong and not flat. To be precise, we consider the following base on $su(2,1)$

$$(3.12) \quad \begin{aligned} Z &\doteq i(E_1^1 + E_3^3 - 2E_2^2); & X &\doteq i(E_1^1 - E_3^3); \\ W &\doteq i(E_1^3 - E_3^1); & Y &\doteq E_1^3 + E_3^1; \\ U &\doteq E_1^2 - E_2^1; & V &\doteq i(E_1^2 + E_2^1); \\ S &\doteq E_2^3 + E_3^2; & T &\doteq i(E_2^3 - E_3^2). \end{aligned}$$

A hyper-paracomplex structure on $SU(2,1)$ is given by

$$(3.13) \quad \begin{aligned} J_3(Z) &\doteq X; & J_3(Y) &\doteq W; & J_2(Z) &\doteq W, & J_2(X) &\doteq Y; \\ J_3(U) &\doteq V; & J_3(S) &\doteq T; & J_2(U) &\doteq T; & J_2(V) &\doteq S, \end{aligned}$$

We claim that the neutral metric g determined by the following orthonormal basis

$$(3.14) \quad \begin{aligned} g(Z, Z) &= g(X, X) = g(U, U) = g(V, V) = 1, \\ g(Y, Y) &= g(W, W) = g(S, S) = g(T, T) = -1 \end{aligned}$$

is a non strong left-invariant HPKT-metric on $SU(2,1)$ with respect to the left-invariant hyper-paracomplex structure (3.13). We denote the 1-form dual to a vector field via the neutral metric (3.14) by the same letter. We calculate

$$\begin{aligned} T^\nabla &= d_1 F_1 = d_2 F_2 = d_3 F_3 \\ &= 2X \wedge Y \wedge W - X \wedge U \wedge V + X \wedge S \wedge T + Y \wedge U \wedge S \\ &\quad - Y \wedge V \wedge T + W \wedge U \wedge T + W \wedge V \wedge S - Z \wedge U \wedge V - Z \wedge S \wedge T; \\ dT^\nabla &= -4U \wedge V \wedge S \wedge T \neq 0. \end{aligned}$$

Our claim is proved.

3.2. **HPKT-structure on $SL(2m-1, \mathbb{R})$.** Consider the simple Lie group $SL(2m-1, \mathbb{R})$ with the almost hyper-paracomplex structure (3.11) applied to the following base³ of $sl(2m-1, \mathbb{R})$:

$$(3.15) \quad \begin{aligned} Z^j &\doteq E_j^j + E_{2m-j}^{2m-j} - 2E_m^m; & W^j &\doteq E_j^j - E_{2m-j}^{2m-j}; \\ X^j &\doteq E_j^{2m-j} - E_{2m-j}^j; & Y^j &\doteq E_j^{2m-j} + E_{2m-j}^j; \\ U_j^k &\doteq E_j^k - E_k^j; & V_j^k &\doteq E_k^{2m-j} - E_{2m-j}^k; \\ S_j^k &\doteq E_k^{2m-j} + E_{2m-j}^k; & T_j^k &\doteq E_j^k + E_k^j. \end{aligned}$$

²We choose the simplest one in a notational sense. Obviously, any neutral metric on the Lie algebra \mathfrak{g} , which is compatible with a paraquaternionic structure, gives a left invariant metric on the corresponding Lie group G . However there is only one biinvariant metric on a simple group.

³The range of indices is as in formula (3.8).

The structure (3.11) is an integrable left-invariant hyper-paracomplex structure on $SL(2m-1, \mathbb{R}), m > 1$ [20].

A left-invariant neutral metric g , determined by the following orthonormal basis of $sl(2m-1, \mathbb{R})$

$$(3.16) \quad \begin{aligned} g(Z^j, Z^j) &= g(X^j, X^j) = g(U_j^k, U_j^k) = g(V_j^k, V_j^k) = 1, \\ g(Y^j, Y^j) &= g(W^j, W^j) = g(S_j^k, S_j^k) = g(T_j^k, T_j^k) = -1 \end{aligned}$$

is an HPKT-metric on $sl(2m-1, \mathbb{R}), m > 1$ with respect to the hyper-paracomplex structure (3.11), which has not closed torsion 3-form. Indeed, denoting the 1-form dual to a vector field via the neutral metric (3.16) by the same letter, we calculate

$$\begin{aligned} T^\nabla &= d_1 F_1 = d_2 F_2 = d_3 F_3 \\ &= 2X^j \wedge Y^j \wedge W^j - X^j \wedge U_j^k \wedge V_j^k + X^j \wedge S_j^k \wedge T_j^k + Y^j \wedge U_j^k \wedge S_j^k \\ &\quad - Y^j \wedge V_j^k \wedge T_j^k + W^j \wedge U_j^k \wedge T_j^k + W^j \wedge V_j^k \wedge S_j^k + Z^j \wedge U_j^k \wedge T_j^k - Z^j \wedge V_j^k \wedge S_j^k; \\ dT^\nabla &= -8U_j^k \wedge V_j^k \wedge S_j^k \wedge T_j^k \neq 0. \end{aligned}$$

The groups $SL(2m-1, \mathbb{R})$ admit cocompact lattices [8], say Γ . Thus, we arrive at a HPKT-extension of the results in [20].

Theorem 3.3. *The compact manifolds $SL(2m-1, \mathbb{R})/\Gamma$ admit invariant HPKT-structure which are not strong.*

3.3. HPKT-structure on $2\mathbb{R} \oplus sl(2, \mathbb{C})$. We consider the following base on $2\mathbb{R} \oplus sl(2, \mathbb{C})$:

$$(3.17) \quad \begin{aligned} Z &\doteq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X \doteq \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; Y \doteq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; W \doteq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ U &\doteq iZ; \quad V \doteq iX; \quad S \doteq iY; \quad T \doteq iW. \end{aligned}$$

We define an almost hyper-paracomplex structure on the Lie algebra $2\mathbb{R} \oplus sl(2, \mathbb{C}) \cong 2\mathbb{R} \oplus so(3, 1)$ by (3.13) using the base (3.17). It is easy to check that this structure is integrable.

We claim that the neutral metric g determined by the orthonormal basis (3.14) is a strong left-invariant HPKT-metric on the simply connected Lie group G associated to the Lie algebras $2\mathbb{R} \oplus sl(2, \mathbb{C}) \cong 2\mathbb{R} \oplus so(3, 1)$ with respect to the left-invariant hyper-paracomplex structure (3.13). To prove the claim, we denote the 1-form dual to a vector field via the neutral metric (3.14) by the same letter. We obtain

$$T^\nabla = d_1 F_1 = d_2 F_2 = d_3 F_3 = S \wedge dS - Y \wedge dY; \quad dT^\nabla = 0$$

which proves our claim.

Let $\Delta \subset SL(2, \mathbb{C})$ be a cocompact discrete subgroup and let $\Gamma = \mathbb{Z} \times \mathbb{Z} \times \Delta \subset G$. We obtain

Theorem 3.4. *The compact manifold G/Γ admits an invariant strong HPKT-structure.*

4. CHARACTERIZATION OF HPKT-STRUCTURES

In this section, we characterize HPKT-structures in terms of the existence of holomorphic objects. We use ideas from the definite (HKT) case described in [16, 14] and find other compact examples.

4.1. Holomorphic characterization. We recall that the space of paracomplex $(1,0)$ -vectors (resp. $(0,1)$ -vectors) with respect to the paracomplex structure $J_\alpha, \alpha = 1, 2$ is spanned by paracomplex vectors of type $X + \epsilon J_\alpha X$ (resp. $X - \epsilon J_\alpha X$) and the space of complex $(1,0)$ -vectors (resp. $(0,1)$ -vectors) of the complex structure J_3 is spanned as usual by complex vectors of type $X - iJ_3 X$ (resp. $X + iJ_3 X$).

It is easy to check that

- : the 2-form $F_2 - \epsilon F_3$ is of type $(2,0)$ while the 2-form $F_2 + \epsilon F_3$ is of type $(0,2)$ with respect to the paracomplex structure J_1 ;
- : the 2-form $F_3 + \epsilon F_1$ is of type $(2,0)$ while the 2-form $F_3 - \epsilon F_1$ is of type $(0,2)$ with respect to the paracomplex structure J_2 ;
- : the 2-form $F_1 - iF_2$ is of type $(2,0)$ while the 2-form $F_1 + iF_2$ is of type $(0,2)$ with respect to the complex structure J_3 ;

Proposition 4.1. *Let $(M, g, J_a, a = 1, 2, 3)$ be a hyperparahermitian manifold. The following conditions are equivalent:*

- a:** $(M, g, J_a, a = 1, 2, 3)$ is a PHKT manifold;
- b:** $\partial_1(F_2 - \epsilon F_3) = 0$ or equivalently $\bar{\partial}_1(F_2 + \epsilon F_3) = 0$;
- c:** $\partial_2(F_3 + \epsilon F_1) = 0$ or equivalently $\bar{\partial}_2(F_3 - \epsilon F_1) = 0$;
- d:** $\partial_3(F_1 - iF_2) = 0$ or equivalently $\bar{\partial}_3(F_1 + iF_2) = 0$;

Proof. We have

$$\partial_1(F_2 - \epsilon F_3) = \bar{\partial}_1(F_2 + \epsilon F_3) = \frac{1}{2}(dF_2 - d_1 F_3) - \frac{\epsilon}{2}(dF_3 - d_1 F_2).$$

Therefore $\partial_1(F_2 - \epsilon F_3) = 0$, when the real and imaginary parts both vanishes. We calculate

$$\begin{aligned} d_1 F_3 &= -J_1 dJ_1 F_3 = -J_1 d(F_3 \circ J_1) = -J_1 dF_3 = (dF_3 \circ J_1) = (dF_3 \circ J_3 J_2) \\ &= -J_3(dF_3 \circ J_2) = J_2 J_3 dF_3 = J_2 J_3 dJ_3 F_3 = J_2 d_3 F_3. \end{aligned}$$

On the other hand

$$dF_2 = -d(F_2 \circ J_2) = -J_2^2 dJ_2 F_2 = J_2 d_2 F_2.$$

Consequently, the condition $d_1 F_3 = dF_2$ is equivalent to the condition $d_2 F_2 = d_3 F_3$. Therefore the Bismut connection of the parahermitian structure (g, J_2) coincides with Bismut connection of the hermitian structure (g, J_3) . Since $J_1 = J_3 J_2$ then J_1 is parallel with respect to the common connection ∇ . Therefore ∇ is the Bismut connection for (g, J_1) which proves the equivalence of a) and b). In a similar way one completes the proof. \square

Proposition 4.1 implies that the HPKT condition is not preserved by a generic conformal transformation of the metric provided the dimension is at least eight.

In the proof of Proposition 4.1, we also derive

Corollary 4.2. *Suppose F_1, F_2 and F_3 are the Kähler forms of a hyperparahermitian structure. Then the hyperparahermitian structure is HPKT-structure if and only if*

$$d_a F_b = \delta_{ab} T^\nabla - \epsilon_{abc} F_c,$$

where δ_{ab} is the Kroneker symbol and ϵ_{abc} is the totally skew-symmetric Levi-Civita symbol.

Theorem 4.3. *Let $(M, J_a, a = 1, 2, 3)$ be a hyperparacomplex manifold. Then any one of the following three conditions implies the forth:*

- (1) $F_2 + \epsilon F_3$ is a $(0, 2)$ -form with respect to J_1 such that $\bar{\partial}_1(F_2 + \epsilon F_3) = 0$ and $F_2(X, J_2 Y) = g(X, Y)$ is a symmetric non-degenerate bilinear form of neutral signature;
- (2) $F_3 - \epsilon F_1$ is a $(0, 2)$ -form with respect to J_2 such that $\bar{\partial}_2(F_3 - \epsilon F_1) = 0$ and $F_1(X, J_1 Y) = g(X, Y)$ is a symmetric non-degenerate bilinear form of neutral signature;
- (3) $F_1 + i F_2$ is a $(0, 2)$ -form with respect to J_3 such that $\bar{\partial}_3(F_1 + i F_2) = 0$ and $F_3(X, J_3 Y) = g(X, Y)$ is a symmetric non-degenerate bilinear form of neutral signature;
- (4) The structure $(g, J_a, a = 1, 2, 3)$ is a PHKT structure.

Proof. In view of the Proposition 4.1, it suffices to prove that the metric g is hyperparahermitian.

Using the fact that $F_2 + \epsilon F_3$ is of type $(0, 2)$ with respect to J_1 . Since $X + \epsilon J_1 X$ is of type $(1, 0)$ with respect to J_1 , $(F_2 + \epsilon F_3)(X + \epsilon J_1 X, Y) = 0$, for any vectors X, Y . It is equivalent to the identity $F_3(X, Y) = -F_2(J_1 X, Y)$. Then

$$F(X, J_3 Y) = -F_2(J_1 X, J_3 Y) = -F_2(J_1 X, J_1 J_2 Y) = -F_2(X, J_2 Y) = -g(X, Y).$$

So $F_3(J_3 X, J_3 Y) = F_3(X, Y)$ and g is hermitian with respect to J_3 . Since the metric is parahermitian with respect to J_2 and $J_1 = J_3 J_2$, g is parahermitian with respect to J_1 .

Similarly, one get the other assertions. \square

4.2. HPKT structures on compact Nilmanifolds. In this section we construct further examples of homogeneous HPKT-structures, now on some (compact) nilmanifolds.

Let $\{X_1, \dots, X_{2n}, Y_1, \dots, Y_{2n}, Z\}$ be a basis for \mathbb{R}^{4n+1} . Define commutators by: $[X_j, Y_j] = Z$, all others being zero. These commutators give \mathbb{R}^{4n+1} the structure of the Heisenberg Lie algebra \mathfrak{h}_{2n} . Let \mathbb{R}^3 be the three dimensional abelian algebra. The direct sum $\mathfrak{n} = \mathfrak{h}_{2n} \oplus \mathbb{R}^3$ is a 2-step nilpotent algebra whose center is four-dimensional. Fix a basis $\{E_1, E_2, E_3\}$ for \mathbb{R}^3 and consider the following endomorphisms of \mathfrak{n} :

$$(4.18) \quad \begin{aligned} J_2 : X_{2j-1} &\rightarrow Y_{2j}, X_{2j} \rightarrow Y_{2j-1} \quad Z \rightarrow E_2, \quad E_1 \rightarrow -E_3; \\ J_3 : X_{2j-1} &\rightarrow X_{2j}, \quad Y_{2j-1} \rightarrow Y_{2j} \quad Z \rightarrow E_1, \quad E_2 \rightarrow E_3; \\ J_2^2 &= -J_3^2 = \text{identity}, \quad J_1 = J_3 J_2. \end{aligned}$$

Clearly $J_2 J_3 = -J_3 J_2$. The almost complex structure J_3 satisfies the identity $[J_3 \cdot, J_3 \cdot] = [\cdot, \cdot]$ which implies that it is an Abelian almost complex structure on \mathfrak{n} in the sense of [5] and in particular integrable. The next notion seems to be useful

Definition 4.4. The almost paracomplex structure J_2 is said to be *Abelian* if the following identity $[J_2., J_2.] = -[.,.]$ holds.

Applying (2.2) it is easy to check that any Abelian almost paracomplex structure has vanishing Nijenuis tensor and therefore is integrable. It is easy to verify that the almost paracomplex structure J_2 is Abelian on \mathfrak{n} . Consequently, the almost paracomplex structure J_1 is also Abelian. Hence, the structure $J_a, a = 1, 2, 3$ is a left invariant hyperparacomplex structure on the simply connected Lie group N whose Lie algebra is \mathfrak{n} . Consider the invariant metric g on N for which the basis $\{X_j, Y_j, Z, E_a\}$ is orthonormal and $g(X_j, X_j) = g(Z, Z) = g(E_1, E_1) = 1, \quad g(Y_j, Y_j) = g(E_2, E_2) = g(E_3, E_3) = -1$. Clearly, the structure $(g, J_a, a = 1, 2, 3)$ is a left invariant hyperparahermitian structure on N which turns out to be a HPKT since any left invariant $(2,0)$ -form with respect to the complex structure J_3 is ∂_3 -closed due to a result of Salamon [31] and Proposition 4.1. Because N is isomorphic to the product $H_{2n} \times \mathbb{R}^3$ of the Heisenbrg group H_{2n} and the Abelian group \mathbb{R}^3 we have:

Corollary 4.5. *Let Γ be a cocompact lattice in the Heisenberg group H_{2n} and \mathbb{Z}^3 a lattice in \mathbb{R}^3 . The compact Nilmanifold $N/(\Gamma \times \mathbb{Z}^3)$ admits an invariant HPKT-structure.*

4.3. HPKT-structure on $(H_{2n} \times \widetilde{SL(2, \mathbb{R})})/\Gamma$. Based on the above computations, we can also see that there is a left-invariant HPKT-structure on the product of $4n+1$ -dimensional Heisenberg group H_{2n} and the universal cover $\widetilde{SL(2, \mathbb{R})}$ of the simple Lie group $SL(2, \mathbb{R})$. The Lie algebra $sl(2, \mathbb{R})$ has a basis $\{E_1, E_2, E_3\}$ with non-zero brackets given by

$$[E_1, E_2] = E_3, \quad [E_2, E_3] = -E_1, \quad [E_3, E_1] = E_2.$$

The construction of the HPKT-structure on the product $H_{2n} \times \widetilde{SL(2, \mathbb{R})}$ is the same as those in Section 4.2 taking E_1, E_2, E_3 to be the generators of $sl(2, \mathbb{R})$. The integrability of the (non-abelian) almost hyper-paracomplex structure defined by (4.2) as well as the HPKT-compatibility conditions (2.5) can be checked directly using the commutators of the left-invariant vector fields. Denote the left-invariant 1-forms dual to the left invariant vector fields via the metric by the same letters to get

$$T^\nabla = d_1 F_1 = d_2 F_2 = d_3 F_3 = dZ \wedge Z; \quad dT^\nabla = dZ \wedge dZ \neq 0.$$

The last equalities imply that the HPKT-structure is not strong.

Let Γ_1 be a cocompact lattice in the Heisenberg group H_{2n} . The universal cover $\widetilde{SL(2, \mathbb{R})}$ of the Lie group $SL(2, \mathbb{R})$ admits a discrete subgroup Γ_2 such that the quotient space $(\widetilde{SL(2, \mathbb{R})}/\Gamma_2)$ is a compact 3-manifold [26, 29, 32]. Such a space has to be Seifert fibre space [32] and all the quotients are classified in [29]. We obtain

Corollary 4.6. *The compact manifold $(H_{2n} \times \widetilde{SL(2, \mathbb{R})})/(\Gamma_1 \times \Gamma_2)$ admits an invariant non-strong HPKT-structure.*

5. POTENTIAL THEORY

It is well known that a Kähler metric is locally generated by a potential ie a real function μ satisfying $F_3 = -dd_3\mu$. Similarly a paraKähler metric is locally generated by a potential, ie a real function ν satisfying $F_1 = dd_1\nu$ [28].

A function μ is a potential function for a hyperparaKähler manifold (M, g, J_a) if the Kähler forms are equal to

$$(5.19) \quad F_a = J_a^2 dd_a \mu, \quad d_a \mu = -J_a^3 d\mu.$$

In this section, we seek a function that generates all Kähler forms of a HPKT-manifold.

The definition of the operators d_a , paraquaternionic identities (2.1), the compatibility conditions (2.3) and (5.19) imply

$$(5.20) \quad \begin{aligned} d_1 d_2 \mu &= -d_1 J_2 d\mu = J_1 d J_1 J_2 d\mu = J_1 dd_3 \mu = -J_1 F_3 = -d_2 d_1 \mu = dd_3 \mu; \\ d_2 d_3 \mu &= d_2 J_3 d\mu = J_2 d J_1 d\mu = -J_2 dd_1 \mu = -J_2 F_1 = -d_3 d_2 \mu = -dd_1 \mu; \\ d_3 d_1 \mu &= -d_3 J_1 d\mu = J_3 d J_3 J_1 d\mu = J_3 dd_2 \mu = J_3 F_2 = -d_1 d_3 \mu = -dd_2 \mu; \end{aligned}$$

We generalize this concept to HPKT-manifold.

Definition 5.1. Let (M, g, J_a) be a HPKT-structure with Kähler forms F_1, F_2 and F_3 . A possibly locally defined function μ is a potential function for the HPKT structure if

$$(5.21) \quad F_1 = \frac{1}{2} (dd_1 - d_2 d_3) \mu, \quad F_2 = \frac{1}{2} (dd_2 - d_3 d_1) \mu, \quad F_3 = -\frac{1}{2} (dd_3 + d_1 d_2) \mu.$$

In fact any one of the above identities implies the others due to the next

Theorem 5.2. Let (M, g, J_a) be a HPKT-structure with Kähler forms F_1, F_2 and F_3 . Let ∇^{CP} be the complex product connection. A possibly locally defined function μ is a potential function for the HPKT structure if any one of the following identities hold

$$(5.22) \quad F_1 = \frac{1}{2} (dd_1 - d_2 d_3) \mu,$$

$$(5.23) \quad F_2 = \frac{1}{2} (dd_2 - d_3 d_1) \mu,$$

$$(5.24) \quad F_3 = -\frac{1}{2} (dd_3 + d_1 d_2) \mu,$$

$$(5.25) \quad g = \frac{1}{2} (1 - J_1 - J_2 + J_3) (\nabla^{CP})^2 \mu.$$

The torsion 3-form T^∇ is given by $T^\nabla = -\frac{1}{2} d_1 d_2 d_3 \mu$.

Proof. We calculate, using the fact that the complex product connection is a torsion-free and preserves the hyper-paracomplex structure, that

$$\begin{aligned} dd_3 \mu(X, Y) &= -(\nabla_X^{CP} d\mu) J_3 Y + (\nabla_Y^{CP} d\mu) J_3 X; \\ d_1 d_2 \mu(X, Y) &= J_1 dd_3 \mu(X, Y) = (\nabla_{J_1 X}^{CP} d\mu) J_2 Y - (\nabla_{J_1 Y}^{CP} d\mu) J_2 X; \\ g(X, Y) &= -F_3(X, J_3 Y) = \frac{1}{2} (dd_3 + d_1 d_2) \mu = \frac{1}{2} (1 - J_1 - J_2 + J_3) (\nabla^{CP})^2 \mu \end{aligned}$$

Thus, the equivalence of (5.24) and (5.25) is proved.

Similarly one can get the equivalence of (5.22) and (5.25) as well as the equivalence between (5.23) and (5.25).

The formula for the torsion is a consequence of (5.21) and (2.6). \square

Remark 5.3. In the context of a potential, an HPKT structure is hyper-paraKähler if and only if the potential function μ satisfies any of the following four identities

$$\begin{aligned} dd_1\mu &= -d_2d_3\mu, \\ dd_2\mu &= -d_3d_1\mu, \\ dd_3\mu &= d_1d_2\mu, \\ (1 + J_1 + J_2 + J_3)(\nabla^{CP})^2\mu &= 0. \end{aligned}$$

Corollary 5.4. *Let (M, g, J_a) be a HPKT-structure with Kähler forms F_1, F_2 and F_3 . A possibly locally defined function μ is a potential function for the HPKT structure if any one of the following identities hold*

$$\begin{aligned} (5.26) \quad F_2 - \epsilon F_3 &= -2\partial_1 J_2 \bar{\partial}_1 \mu; \\ (5.27) \quad F_3 + \epsilon F_1 &= -2\partial_2 J_3 \bar{\partial}_2 \mu; \\ (5.28) \quad F_1 - i F_2 &= -2\partial_3 J_1 \bar{\partial}_3 \mu. \end{aligned}$$

Proof. Due to (5.20) and the definition of the operators $\partial_a, \bar{\partial}_a$, we have

$$F_1 - i F_2 = \frac{1}{2} (dd_1 - d_2d_3 - idd_2 + id_3d_1) = -2\partial_3 J_1 \bar{\partial}_3 \mu$$

The other assertions follow in a similar way. \square

5.1. HPKT potential in dimension 4. Now we give a hyperbolic version of the existence of the HKT potentials of Banos and Swann [4]. We apply Theorem 5.2 to the 4-dimensional case to prove the existence of a local HPKT potential for any HPKT metric.

Corollary 5.5. *Let g be an HPKT metric on a 4-dimensional hyper-paracomplex manifold and let θ be 1-form defined by the complex product connection via $\nabla^{CP}g = \theta \otimes g$.*

A function μ is an HPKT potential for g if and only if it is solution of the hyperbolic equation

$$\Delta\mu - d\mu(\theta^\sharp) + 2 = 0,$$

where Δ is the hyperbolic Laplacian of the neutral metric g .

In particular, any HPKT metric on a 4-dimensional hyper-paracomplex manifold admits locally a potential.

Proof. Let $A = \nabla^{CP} - \nabla^g$, where ∇^g denote the Levi-Civita connection of g . The tensor A is symmetric, $A(X, Y) = A(Y, X)$ since both connections are torsion-free. We also have

$$\theta(X)g(Y, Z) = -g(A(X, Y), Z) - g(A(X, Z), Y).$$

Solving for A , we obtain

$$g(A(X, Y), Z) = \frac{1}{2} (\theta(Z)g(X, Y) - \theta(X)(g(Y, Z) - \theta(Y)g(Y, Z))).$$

In particular, if X is a (local) unit vector field, then

$$g(A(X, X), Y) = \frac{1}{2} \theta(Y) - \theta(X)g(X, Y)$$

and

$$g(A(X, X) - A(J_1 X, J_1 X) - A(J_2 X, J_2 X) + A(J_3 X, J_3 X), Y) = \theta(Y)$$

for all Y .

The metric g is the unique hyper-parahermitian metric satisfying $g(X, X) = 1$. Therefore μ is a HPKT potential if and only if

$$\frac{1}{2} (1 - J_1 - J_2 + J_3) (\nabla^{CP})^2(X, X) = 1,$$

that is

$$\text{Trace}_g(\nabla^{CP} d\mu) = 2.$$

Note that the hyperbolic Laplacian $\Delta\mu$ is by definition $-\text{Trace}_g(\nabla^g d\mu)$. Thus μ is a HPKT potential for g if and only if

$$-\Delta\mu + d\mu(A(X, X) - A(J_1 X, J_1 X) - A(J_2 X, J_2 X) + A(J_3 X, J_3 X)) = 2.$$

The local existence of HPKT potentials now follows from the general theory for the (ultra) hyperbolic Laplace operator (see e.g. [17] and the references therein). \square

5.2. HPKT-potential in dimension $4n \geq 8$. Here we demonstrate the existence of a local potential for any HPKT-metric (the HKT case was done by Banos and Swann [4]).

The crucial step is the construction on any almost hyperparacomplex manifold of an $GL(n, \tilde{H})$ -invariant first order differential operator D , which is the hyperbolic version of the hypercomplex (quaternionic) operator of Salamon (see [30]). The operator D is two-step nilpotent if and only if the structure is hyperparacomplex. Then we obtain the existence of a local HPKT-potential in terms of the operator D .

The element

$$\dagger = -J_1 \otimes J_1 - J_2 \otimes J_2 + J_3 \otimes J_3$$

is independent of the choice of the basis $\{J_1, J_2, J_3\}$ and acts naturally on Λ^2 with $\dagger^2 = 2\dagger + 3$. The eigenspace decomposition

$$(5.29) \quad \Lambda^2 = \{\dagger = 3\} \oplus \{\dagger = -1\}$$

is a paraquaternionic invariant in the sense that it is preserved by $GL(n, \tilde{H})Sp(1, \mathbb{R})$ and therefore it is a hyper-paracomplex invariant preserving by $GL(n, \tilde{H})$.

5.3. The hyper-paracomplex differential. Studying the action of $GL(n, \tilde{H})$ on the bundle Λ^k we consider the subbundle

$$A^k = \sum_{I \in S_1^2(-1)} \left(\Lambda_I^{k,0} \oplus \Lambda_I^{0,k} \right).$$

It is not difficult to see that

$$A^k = \sum_{P \in S_1^2(1)} \left(\Lambda_P^{k,0} \oplus \Lambda_P^{0,k} \right).$$

Indeed, any 2-form $\omega \in \Lambda^2$ decomposes according to (5.29)

$$(5.30) \quad \begin{aligned} \omega(X, Y) &= \frac{1}{4} \{ 3\omega(X, Y) + \omega(J_1 X, J_1 Y) + \omega(J_2 X, J_2 Y) - \omega(J_3 X, J_3 Y) \} \\ &\quad + \frac{1}{4} \{ \omega(X, Y) - \omega(J_1 X, J_1 Y) - \omega(J_2 X, J_2 Y) + \omega(J_3 X, J_3 Y) \} \end{aligned}$$

For example

$$A^2 = \Lambda_{J_3}^{2,0} \oplus \Lambda_{J_3}^{0,2} \oplus A_{J_3}^{1,1} = \Lambda_{J_2}^{2,0} \oplus \Lambda_{J_2}^{0,2} \oplus A_{J_2}^{1,1},$$

where

$$\begin{aligned} A_{J_3}^{1,1} &= \{ \omega \in \Lambda^2 : J_3 \omega = \omega \quad \text{and} \quad J_2 \omega = \omega \}, \\ A_{J_2}^{1,1} &= \{ \omega \in \Lambda^2 : J_2 \omega = -\omega \quad \text{and} \quad J_3 \omega = -\omega \}. \end{aligned}$$

If g is an hyper-parahermitian metric then the Kähler form F_3 , (resp. F_2) is a smooth section of $A_{J_3}^{1,1}$, (resp. $A_{J_2}^{1,1}$) and conversely any smooth section F_3 of $A_{J_3}^{1,1}$, (resp. F_2 of $A_{J_2}^{1,1}$) defines an (possibly degenerate) hyper-parahermitian metric $g = -F_3(\cdot, J_3 \cdot)$, (resp. $g = F_2(\cdot, J_2 \cdot)$). We will call such a form a *hyper-paracomplex (1,1)-form*.

There is a projection $\eta : \Lambda^k \rightarrow A^k$ whose kernel is the subbundle

$$\begin{aligned} B^k &= \bigcap_{I \in S_1^2(-1)} \left(\Lambda_I^{k-1,1} \oplus \Lambda_I^{k-2,2} \oplus \dots \oplus \Lambda_I^{1,k-1} \right) = \\ &= \bigcap_{P \in S_1^2(1)} \left(\Lambda_P^{k-1,1} \oplus \Lambda_P^{k-2,2} \oplus \dots \oplus \Lambda_P^{1,k-1} \right). \end{aligned}$$

In particular, the two eigenspaces of the operator \dagger are related with A^2, B^2 as follows

$$A^2 = \{ \dagger = -1 \}, \quad B^2 = \{ \dagger = 3 \}.$$

The projections ω^{A^2} and ω^{B^2} are given by

$$\begin{aligned} \omega^{A^2}(X, Y) &= \frac{1}{4} \{ 3\omega(X, Y) + \omega(J_1 X, J_1 Y) + \omega(J_2 X, J_2 Y) - \omega(J_3 X, J_3 Y) \}, \\ \omega^{B^2}(X, Y) &= \frac{1}{4} \{ \omega(X, Y) - \omega(J_1 X, J_1 Y) - \omega(J_2 X, J_2 Y) + \omega(J_3 X, J_3 Y) \}. \end{aligned}$$

We define the *hyper-paracomplex differential*

$$D : A^k \rightarrow A^{k+1}$$

simply by composition of the projection η and the exterior differential d :

$$D = \eta \circ d.$$

For example, if ω is a 1-form then

$$\begin{aligned} (5.31) \quad D\omega &= (d\omega)_{J_3}^{2,0} + (d\omega)_{J_3}^{0,2} + \frac{1}{2} \left((d\omega)_{J_3}^{1,1} + J_2(d\omega)_{J_3}^{1,1} \right) = \\ &= (d\omega)_{J_2}^{2,0} + (d\omega)_{J_2}^{0,2} + \frac{1}{2} \left((d\omega)_{J_2}^{1,1} - J_3(d\omega)_{J_2}^{1,1} \right) \end{aligned}$$

Theorem 5.6. *An almost hyper-paracomplex structure is integrable if and only if $D^2 = 0$.*

Proof. The condition $D^2 = 0$ is equivalent to the assertion that the exterior derivative of a 2-form of type (1,1) relative to all $I \in S_1^2(-1)$ and all $P \in S_1^2(1)$ has no (0,3)+(3,0)-component relative to any $I \in S_1^2(-1)$ and to any $P \in S_1^2(1)$. The latter condition holds on a hyper-paracomplex manifold since all almost complex structures of the two-sheeted hyperboloid $S_1^2(-1)$ as well as all paracomplex structures of the one-sheeted hyperboloid $S_1^2(1)$ are integrable due to the existence of a linear identity between their Nijenhuis tensors [21, 7].

To prove the converse, let Ω be (1,1)-form with respect to the almost complex structure $J_3, \Omega \in \Lambda_{J_3}^{1,1}$. The 2-form C defined by $C(X, Y) = \Omega(X, Y) - \Omega(J_1 X, J_1 Y)$ belongs to B^2 . The condition $DC = 0$ is equivalent to the relation

$$dC(X, Y, Z) = dC(IX, IY, Z) + dC(IX, Y, IZ) + dC(X, IY, IZ)$$

for all $I \in S_1^2(-1)$ and

$$-dC(X, Y, Z) = dC(PX, PY, Z) + dC(PX, Y, PZ) + dC(X, PY, PZ)$$

for all $P \in S_1^2(1)$. In particular

$$\begin{aligned} (5.32) \quad & -dC(J_3 X, J_3 Y, J_3 Z) = \\ & dC(J_1 X, J_1 Y, J_3 Z) + dC(J_1 X, J_3 Y, J_1 Z) + dC(J_3 X, J_1 Y, J_1 Z). \end{aligned}$$

Let ∇^{CP} be a complex product connection, $\nabla^{CP} J_a = 0$, with torsion T^{CP} . Then the Nijenhuis tensors are related with T^{CP} as follows

$$(5.33) \quad N_a = -T^{CP}(J_a, J_a) - J_a^2 T^{CP}(.,.) + J_a T^{CP}(J_a, .) + J_a T^{CP}(., J_a).$$

Use (5.33) to express the exterior derivative of a 2-form as

$$\begin{aligned} (5.34) \quad dC(X, Y, Z) &= \nabla^{CP} C(X; Y, Z) + \nabla^{CP} C(Y; Z, X) + \nabla^{CP} C(Z; X, Y) \\ &+ C(T^{CP}(X, Y), Z) + C(T^{CP}(Y, Z), X) + C(T^{CP}(Z, X), Y). \end{aligned}$$

Insert (5.34) into (5.32) and use (5.33) to get

$$\begin{aligned} (5.35) \quad & \Omega(J_3 N_2(J_3 X, J_3 Y), Z) + \Omega(J_3 N_2(J_3 Y, J_3 Z), X) + \Omega(J_3 N_2(J_3 Z, J_3 X), Y) + \\ & \Omega(J_1 N_2(J_3 X, J_3 Y), J_2 Z) + \Omega(J_1 N_2(J_3 Y, J_3 Z), J_2 X) + \Omega(J_1 N_2(J_3 Z, J_3 X), J_2 Y) = 0 \end{aligned}$$

valid for any (1,1)-form with respect to J_3 . In particular, take $\Omega = Z \wedge J_3 Z$, we get from (5.35) that $N_2 = 0$.

Similarly, we obtain $N_1 = 0$. Hence, the almost hyper-paracomplex structure is integrable [21]. \square

Further we need two lemmas

Lemma 5.7. *Let $F \in A_{J_3}^{1,1}$ be a non-degenerate hyperparacomplex $(1,1)$ -form on a hyper-paracomplex manifold (M, J_1, J_2, J_3) . The metric $g = -F(., J_3.)$ is HPKT metric if and only if F is D -closed, $DF = 0$.*

Such a form is called a HPKT-form

Proof. Suppose that g is HPKT. For any complex structure $I \in S_1^2(-1)$ (paracomplex structure $P \in S_1^2(1)$) the form dF_I (dF_P) has type $(2,1) + (1,2)$ with respect to the complex structure I (paracomplex structure P). But since $d_1F_1 = d_2F_2 = d_3F_3$ we deduce that dF_I (dF_P) has type $(2,1) + (1,2)$ with respect to the two paracomplex structures and the complex structure: $dF_I \in B^3$ ($dF_P \in B^3$) that is $DF_I = 0$ ($DF_P = 0$). Since $F_I = F$ ($F_P = F$) we obtain the result.

Suppose now that $DF = 0$. The condition $F \in A_{J_3}^{(1,1)}$ also reads as

$$(5.36) \quad F(X, Y) = F(J_3X, J_3Y) = F(J_1X, J_1Y), F(J_3X, J_1Y) = F(J_1X, J_3Y).$$

Use the torsion free complex product connection, (5.34), (5.36) and (5.32), to get

$$\begin{aligned} -dF(J_3X, J_3Y, J_3Z) &= \nabla^{CP}F(J_3X; Y, Z) + \nabla^{CP}F(J_3Y; Z, X) + \nabla^{CP}F(J_3Z; X, Y) \\ &\quad + 2\nabla^{CP}F(J_1X; J_1Y, J_3Z) + 2\nabla^{CP}F(J_1Y; J_1Z, J_3X) + 2\nabla^{CP}F(J_1Z; J_1X, J_3Y). \end{aligned}$$

Consequently

$$\begin{aligned} -dF(J_3X, J_3Y, J_3Z) &= \\ \nabla^{CP}F(J_1X; J_1Y, J_3Z) &+ \nabla^{CP}F(J_1Y; J_1Z, J_3X) + \nabla^{CP}F(J_1Z; J_1X, J_3Y). \end{aligned}$$

Define $G = F(., J_2.)$, we have the following sequence of equalities

$$\begin{aligned} -dG(J_1X, J_1Y, J_1Z) &= \\ -\nabla^{CP}G(J_1X; J_1Y, J_1Z) &- \nabla^{CP}G(J_1Y; J_1Z, J_1X) - \nabla^{CP}G(J_1Z; J_1X, J_1Y) = \\ = \nabla^{CP}F(J_1X; J_1Y, J_3Z) &+ \nabla^{CP}F(J_1Y; J_1Z, J_3X) + \nabla^{CP}F(J_1Z; J_1X, J_3Y) = \\ = -dF(J_3X, J_3Y, J_3Z) \end{aligned}$$

and therefore $dF(J_3X, J_3Y, J_3Z) = dG(J_1X, J_1Y, J_1Z)$. In other words, $d_3F_3 = d_1F_1$ with $F_3 = F$ and $F_1 = G$. \square

Lemma 5.8. *A PHKT metric locally admits a potential if and only if the corresponding HPKT-form is locally D -exact.*

Proof. Suppose that $F = -\frac{1}{2}(dd_3 + d_1d_2)\mu$. Then $F = \frac{1}{2}(d\theta + J_1d\theta)$ with $\theta = -J_3d\mu = -d_3\mu$. Note that $d\theta$ is $(1,1)$ -form (for J_3) since $d\theta = -dd_3\mu$. Therefore $F = D\theta$ according to (5.31).

Conversely, suppose that $F = D\theta$ for some 1-form θ . Since F is a $(1,1)$ -form for J_3 , we obtain from (5.31)

$$d\theta \in \Lambda_{J_3}^{(1,1)}, \quad F = \frac{1}{2}(d\theta + J_1d\theta).$$

Since J_3 is an integrable complex structure, the local dd_3 -lemma holds: locally there exists μ such that $d\theta = -dd_3\mu$. We get then

$$F = \frac{1}{2}(d\theta + J_1 d\theta) = -\frac{1}{2}(dd_3 + d_1 d_2)\mu.$$

□

Theorem 5.9. *On an $(4n \geq 8)$ -dimensional manifold any HPKT metric admits locally an HPKT potential or equivalently any D -closed HPKT-form is locally D -exact.*

Proof. Any $(4n \geq 8)$ -dimensional HPKT-manifold is a manifold with a structure group contained in $Sp(n, \mathbb{R}) \subset GL(n, \tilde{H})$ and it is 1-integrable due to the existence of a torsion-free $GL(n, \tilde{H})$ -connection, the complex product connection [2, 3]. Therefore, since the operator D is $GL(n, \tilde{H})$ -invariant, it is sufficient to show that any D -closed HPKT-form is locally D -exact in \mathbb{R}^{4n} with the standard HPKT structure.

In \mathbb{R}^{4n} we split the complex coordinates into two sets $(\{z^j, w^j\}, j = 1, \dots, n)$. The hyperparacomplex structure is given by

$$\begin{aligned} J_1 &= -dz^j \otimes \frac{\partial}{\partial z^j} - dw^j \otimes \frac{\partial}{\partial w^j} + d\bar{z}^j \otimes \frac{\partial}{\partial \bar{z}^j} + d\bar{w}^j \otimes \frac{\partial}{\partial \bar{w}^j}, \\ J_2 &= id\bar{w}^j \otimes \frac{\partial}{\partial z^j} - idz^j \otimes \frac{\partial}{\partial w^j} + idw^j \otimes \frac{\partial}{\partial \bar{z}^j} - idz^j \otimes \frac{\partial}{\partial \bar{w}^j}, \\ J_3 &= id\bar{w}^j \otimes \frac{\partial}{\partial z^j} - id\bar{z}^j \otimes \frac{\partial}{\partial w^j} - idw^j \otimes \frac{\partial}{\partial \bar{z}^j} + idz^j \otimes \frac{\partial}{\partial \bar{w}^j}. \end{aligned}$$

Let $F_3 \in \Lambda_{J_3}^{1,1}$, $DF_3 = 0$. The hyper-parahermitian condition for the 2-tensor $h = F_3(\cdot, J_3)$ implies that

$$(5.37) \quad h_{z^j \bar{z}^k} = h_{w^k \bar{w}^j}; \quad h_{z^j \bar{w}^k} = -h_{z^k \bar{w}^j}.$$

The condition $d_1 F_1 = d_3 F_3$ becomes

$$\begin{aligned} (5.38) \quad & h_{z^j \bar{w}^k, \bar{w}^l} + h_{z^k \bar{w}^l, \bar{w}^j} + h_{z^l \bar{w}^j, \bar{w}^k} = 0, \quad h_{w^j \bar{z}^k, w^l} + h_{w^k \bar{z}^l, w^j} + h_{w^l \bar{z}^j, w^k} = 0 \\ & h_{w^j \bar{z}^k, \bar{z}^l} + h_{w^k \bar{z}^l, \bar{z}^j} + h_{w^l \bar{z}^j, \bar{z}^k} = 0, \quad h_{z^j \bar{w}^k, z^l} + h_{z^k \bar{w}^l, z^j} + h_{z^l \bar{w}^j, z^k} = 0, \\ & h_{z^j \bar{z}^l, \bar{w}^k} - h_{z^k \bar{z}^l, \bar{w}^j} - h_{z^j \bar{w}^k, \bar{z}^l} = 0, \quad h_{z^j \bar{z}^k, w^l} - h_{z^j \bar{z}^l, w^k} + h_{w^k \bar{z}^l, z^j} = 0, \\ & h_{z^j \bar{z}^k, \bar{z}^l} - h_{z^j \bar{z}^l, \bar{z}^k} - h_{w^k \bar{z}^l, \bar{w}^j} = 0, \quad h_{z^j \bar{z}^l, z^k} - h_{z^k \bar{z}^l, z^j} + h_{z^j \bar{w}^k, w^l} = 0. \end{aligned}$$

The first and second lines of (5.38), when combined with the antisymmetry in j, k of $h_{z^j \bar{w}^k}$, allow us to apply the local $\partial\bar{\partial}$ -lemma. Therefore, we can write

$$(5.39) \quad h_{z^j \bar{w}^k} = (\partial_{z^j} \partial_{\bar{w}^k} - \partial_{z^k} \partial_{\bar{w}^j})\mu; \quad h_{w^j \bar{z}^k} = (\partial_{w^j} \partial_{\bar{z}^k} - \partial_{w^k} \partial_{\bar{z}^j})\mu,$$

where μ is some (real) (by para-hermiticity of the metric-and therefore identical in the two equations (5.39)) function. Inserting (5.39) into the third equation of (5.38) gives

$$(5.40) \quad \partial_{\bar{w}^k}(h_{z^j \bar{z}^l} - \mu_{, z^j \bar{z}^l}) - \partial_{\bar{w}^j}(h_{z^k \bar{z}^l} - \mu_{, z^k \bar{z}^l}) = 0,$$

and therefore,

$$(5.41) \quad h_{z^j \bar{z}^k} = \mu_{, z^j \bar{z}^k} + \partial_{\bar{w}^j} \alpha_{\bar{z}^k}$$

for some integration one form $\alpha_{\bar{z}^k}$. Combining this with the fourth equation of (5.38) gives $\alpha_{\bar{z}^k} = \mu_{,w^k}$. Thus we get that the function μ generates F_3 . The Lemma 5.8 completes the proof. \square

Remark 5.10. A hyperbolic version of Salamon's quaternionic operator [30]. We recall that an *almost paraquaternionic structure* on M is a rank-3 subbundle $P \subset \text{End}(TM)$ which is locally spanned by an almost hyper-paracomplex structure $\tilde{H} = (J_a)$. Equivalently, the structure group of TM can be reduced to $GL(n, \tilde{H})Sp(1, \mathbb{R})$. A linear connection on TM is called *paraquaternionic connection* if it preserves P . An almost paraquaternionic structure is said to be a *paraquaternionic* if there is a torsion-free paraquaternionic connection. The paraquaternionic condition controls the Nijenhuis tensors in the sense that $N(X, Y)(J_a) := N_a(X, Y)$ preserves the subbundle P . When $n \geq 2$, the paraquaternionic condition is a strong condition which is equivalent to the 1-integrability of the associated $GL(n, \tilde{H})Sp(1, \mathbb{R})$ -structure [2, 3]. We can extend the hyper-paracomplex operator D defining it on an almost paraquaternionic manifold locally in the same way. Consequently, Theorem 5.6 is also true, namely an almost paraquaternionic manifold is paraquaternionic exactly when $D^2 = 0$. The proof of Theorem 5.6 goes through in this case also. Now the $Sp(1, \mathbb{R})$ -part of the paraquaternionic connection used, adds an additional $Sp(1, \mathbb{R})$ term in formula (5.33) which reflects on (5.35), whence the Nijenhuis tensors preserve the subbundle P . Using the 1-integrability of the paraquaternionic structure and the proof of Theorem 5.9 one gets the local exactness of certain D -closed forms.

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